Cheating Husbands Puzzles - A Study of the Interplay between Knowledge, Action and Communication

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(Based on a 1986 paper in Distributed Computing by Moses, Dolev, and Halpern)

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Understanding the subtle relationship between knowledge, action, and communication is fundamental to the design of computer network protocols, intelligent robots, etc.

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It involves an initial step in which a set of facts is announced publicly, thereby becoming common knowledge.

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A close analysis of her account illustrates how an agent that knows something about how other individuals' actions are related to the facts they know, can obtain knowledge by observing the other individuals' actions.

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It was also common knowledge that all women hear every gunshot fired in Mamajorca.

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"There are (one or more) unfaithful husbands in our community. Although none of you knew before this gathering whether your own husband was faithful, each of you knows which of the other husbands are unfaithful. I forbid you to discuss the matter of your husband's fidelity with anyone. However, should you discover that your husband is unfaithful, you must shoot him on the midnight of the day you find out about it."

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This leaves us with the following unanswered questions:

1) How many unfaithful husbands were in Mamajorca at the time?

2) How did some cheated wives learn of their husbands' infidelity after thirty nine nights in which nothing happened?

3) Were any more husbands shot on later nights (after the 40th)?

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His wife, upon hearing the queen's statement, would have concluded that her own husband was unfaithful, and would have shot him on the midnight of the first night!

Hence, there must have been more than one unfaithful husband.

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Then, every cheated wife would have initially known of exactly one unfaithful husband, and would have reasoned as follows: "If the unfaithful husband I know of is the only unfaithful husband, then his wife will shoot him on the first night."

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On the morning of the second day each cheated wife would have realized that the unfaithful husband she knew about was not the only one, and that therefore her own husband must be unfaithful.

Thus, both the unfaithful husbands would have been shot on the second night.

A Deeper Analysis

Similar reasoning is used by the wives in general, which leads us to the following theorem:

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We wish to show that, if there were $n = k + 1$ unfaithful husbands, they would have been shot on the $(k + 1)$-th night.

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Hence, all the unfaithful husbands are shot on the $(k+1)$-th night, and the theorem follows by induction.

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It follows that after the queen speaks, it is common knowledge that there is at least one unfaithful husband.

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Given the wives' famous logical capabilities, it is common knowledge that if there is only one unfaithful husband then he will be shot on the first night.

Therefore, once the first night is silent, it becomes common knowledge that there are at least two unfaithful husbands.

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So although a wife that knows of k unfaithful husbands knows that there will be no shots before the k-th night, her state of knowledge changes following every silent night, even though there is no explicit communication at all!

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Does allowing the wives to communicate a small amount of extra information allow a substantially faster solution to the problem?

Thank You!